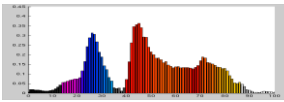


Signals and Systems

Lecture 8

Time Domain Models of Systems

Outline

- **Convolution Model.**
- **Convolution of D-T signals.**
- **Unit Step Response $s[n]$.**
- **Properties of discrete-time linear convolution.**

Convolution Model

Convolution is the most important and fundamental concept in signal processing and analysis. By using convolution, we can construct the output of system for any arbitrary input signal, if we know the impulse response of system.

To represent D-T systems using the convolution model (input-output representation), we need to explain two terms:

1. **Unit Impulse Response $h[n]$:**
2. **Unit Step Response $s[n]$:**

Unit Impulse Response $h[n]$:

This term is used to represent the output of the LTI D-T system when the input, $x[n]$, is the unit impulse $\delta[n]$ and denoted as $h[n]$, the block diagram of an LTI system with impulse input is shown in **figure 4-2**.

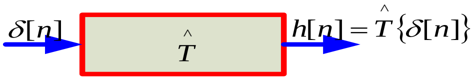


Figure 4-2 Block Diagram of LTI System using Convolution Model

The system is time invariant, so we can write the following:

$$h[n-k]=\hat{T}\{\delta[n-k]\}$$

To compute the response of unit impulse simply insert $x[n]=\delta[n]$ in the general form to get the following equation:

$$h[n]=\sum_{k=0}^{n} w_k\delta[n-k], \quad n\geq 0, \text{ where } h[n]=y[n]$$

Since $\delta[n-k]=1$ when $i=n$, the previous equation can be reduced to

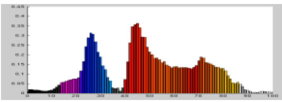
$$h[n]=w_n, \quad n\geq 0$$

This means that the impulse response of the system at time n is equal to w_n

For example, in tables 1 the Impulse Response' Coefficients of $15-$point MA and $15-$point EWMA filters are calculated setting the equation $h[n]=w_n, n=0,1,2,\ldots,14$ and $h[n]=0$ for $n>14$ and $n<0$.

Table 1: Impulse Response' Coefficients of MA filter $h[n]$

Index	$h[0]$	$h[1]$	$h[2]$	$h[3]$	$h[4]$	$h[5]$	$h[6]$	$h[7]$	$h[8]$	$h[9]$	$h[10]$	$h[11]$	$h[12]$	$h[13]$	$h[14]$
Coefficient	.066	.066	.067	.067	.067	.067	.067	.067	.067	.067	.067	.067	.067	.067	.067



Impulse Response' Coefficients of EWMA filter $h[n]$															
Index	$h[0]$	$h[1]$	$h[2]$	$h[3]$	$h[4]$	$h[5]$	$h[6]$	$h[7]$	$h[8]$	$h[9]$	$h[10]$	$h[11]$	$h[12]$	$h[13]$	$h[14]$
Coefficient	.301	.211	.147	.103	.072	.051	.036	.025	.017	.012	.008	.006	.004	.002	.002

Note: the unit impulse response of $15-$point MA and $15-$point EWMA filters are finite duration signals, the response is nonzero for only a finite number of values n

Rewriting $y[n]=\sum_{k=0}^{n} w_k x[n-k], \quad n\geq 0$ in terms of the unit impulse response gives:

$$y[n]=\sum_{k=0}^{n} h[k]x[n-k], \quad n\geq 0$$

This equation is a important equation in designing and analysis of D-T systems and called the **Convolution Sum** of $h[n]$ and $x[n]$, and denoted by the symbol "$*$", we can represent the system by

$$y[n]=h[n]*x[n]=\sum_{k=0}^{n} h[k]x[n-k], \quad n\geq 0$$

The detail analysis of the above equation can be done using the following example (see figure 4-3):

Convolution of D-T signals

The impulse response of a system can be used to determine the response to any input, the representation of discrete time signals in terms of impulses. The key idea is to express an arbitrary discrete time signal as weighted sum of time shifted impulses (values of the samples $x[k]$) and shifted unit impulses $\delta[n-k]$ like that:

$$x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

The response to an arbitrary input $x[n]$ is given by:

$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]=x[n]*h[n]$$ (Discrete version of convolution)

The convolution operation is defined for an arbitrary D-T signals $v[n]$ and $w[n]$ that are not necessary zero for $n<0$, the equation for this case (general case)will be as the following:

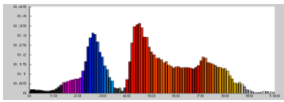
$$y[n]=v[n]*w[n]=\sum_{k=-\infty}^{\infty} v[k]w[n-k]$$

In some special cases the nature of the signals that will be convolved can reduce the limit of the summation, an example of these cases, is the case when the input signals $v[n]$ and $w[n]$ are zero for all samples less than zero ($n<0$) then

$v[n]=0$ for $n<0$; $w[n-k]=0$ for $n<k$;

The convolution summation will be

$$y[n]=v[n]*w[n]=\begin{cases} 0 & n=-1,-2,-3,\ldots \\ \sum_{k=0}^{n} v[k]w[n-k] & n=0,1,2,\ldots \end{cases}$$



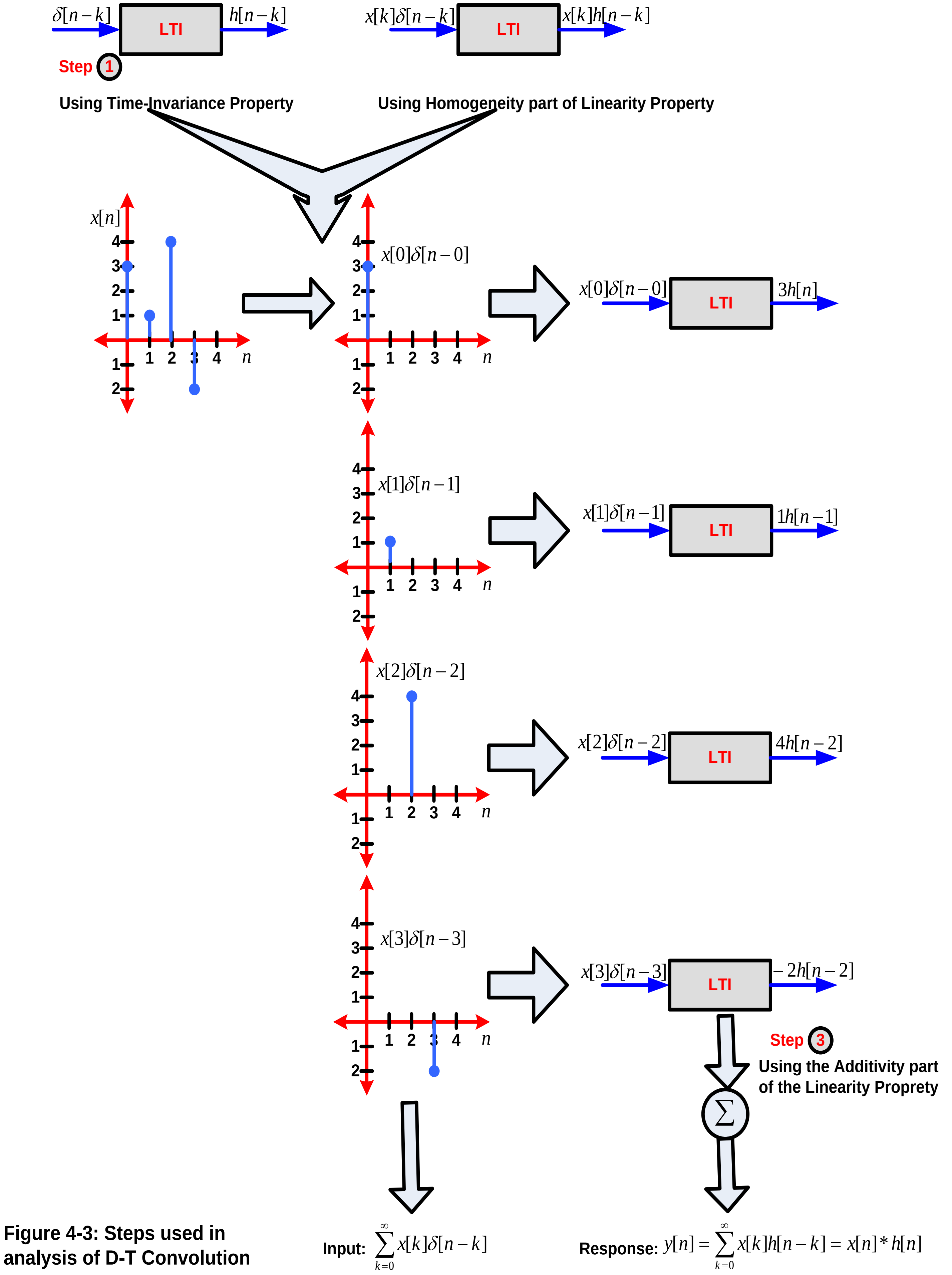
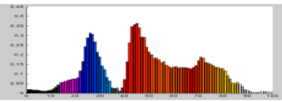


Figure 4-3: Steps used in analysis of D-T Convolution



Impulse Response sources:

We can form the impulse response using different possible ways:

- It can be given by the **designer of the D-T system**.
- It can be **measured experimentally** (you can put in the input a testing sequence of samples and see what comes at the output of the system).
- It can be **mathematically** analyzed for concrete system.

Unit Step Response $s[n]$

Step response is the response of a D-T linear system to a unit step input and denoted as $s[n]=\hat{T}\{u[n]\}$.

In **figure 4-4**, the block diagram of the unit step input passing through an LTI system.

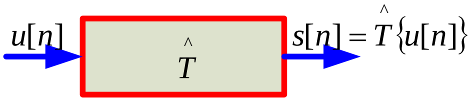


Figure 4-4: Step Response

To compute the step response of the system, by the definition of an LTI system just replace instead any input with the unit step signal, we will obtain the **convolution of unit step signal with the impulse response** of the system defined by the following equation:

$$s[n]=h[n]*u[n]=\sum_{k=-\infty}^{\infty}u[k]h[n-k]$$

The convolution as we will prove later is a **commutative**:

$$s[n]=h[n]*u[n]=u[n]*h[n]$$

$$=\sum_{k=-\infty}^{\infty}u[k]h[n-k]=\sum_{k=-\infty}^{\infty}h[k]u[n-k]=\sum_{k=-\infty}^{n}h[k]$$

Relationship between $h[n]$ and $s[n]$

The relationship between the step response and the impulse response is exists depending on the relationship between unit impulse and unit step, so if we know the step response of a system, we can find the impulse response.

Example: To find the step response for a system with impulse response $h[n]=c\delta[n]+\delta[n-m]$, we go through the following steps:

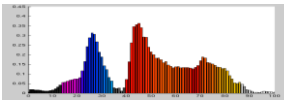
- Using the convolution definition of the unit step response with impulse response, we can write

$$s[n]=h[n]*u[n]=\sum_{k=-\infty}^{\infty}h[n-k]u[k],$$

The limit and the argument of the summation can be reduced because of the nature of the unit step signal ($u[k]=0$ for $k<0$, and 1 for $k\geq 0$), then the equation becomes as the following:

$$s[n]=\sum_{k=0}^{\infty}h[n-k]u[k]=\sum_{k=0}^{\infty}h[n-k]$$

- Substitute the shifted version of $h[n]=c\delta[n]+\delta[n-m]$ by k (shift to the right), we obtain



$$s[n]=\sum_{k=0}^{\infty} c\delta[n-k]+\delta[n-m-k]$$

- Using the relationship between the unit step and the unit impulse

$$u[n]=\sum_{k=-\infty}^{n}\delta[k]$$

We obtain

First tem: $\sum_{k=0}^{\infty} c\delta[n-k]=cu[n]$ and second term: $\sum_{k=0}^{\infty}\delta[n-m-k]=u[n-m]$

- The final result : $s[n]=cu[n]+u[n-m]$

Properties of discrete-time linear convolution

The convolution operation satisfies the following properties:

1. Commutative: $x[n]*h[n]=h[n]*x[n]$

We can interchange the role of $x[n]$ and $h[n]$, which means that we can interpret $x[n]$ as the impulse response of the system and $h[n]$ as the excitation or the input signal, the **figure 4-5** explains this case .

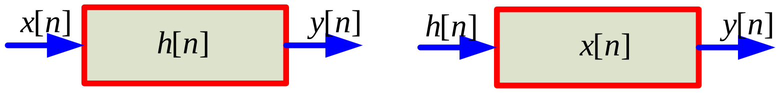


Figure 4-5: Interpretation of the commutative property of convolution

commutative property implies that we can interchange input and impulse response of the LTI system

2. Associative: $(x[n]*h_1[n])*h_2[n]=x[n]*(h_1[n]*h_2[n])$

This can be interpreted as **cascade connection** of LTI systems, which is means that $x[n]$ is the input signal to an LTI system with $h_1[n]$, the output of this system lets denote as $w_1[n]$ becomes the input to a second LTI system with $h_2[n]$, then the output is :

$$y[n]=w_1[n]*h_2[n]=(x[n]*h_1[n])*h_2[n]$$

The result equation for LTI systems for a cascade connection will be

$$y[n]=x[n]*h[n]$$

Where

$$h[n]=h_1[n]*h_2[n]$$

The generalization of associative property to more than two systems in series follows directly by mathematical induction as the following

$$h[n]=h_1[n]*h_2[n]*...*h_k[n]$$

The associative property implies that series connection of two LTI systems is an LTI system. Where impulse response is convolution of individual responses. The commutative property implies that we can interchange the order of the two systems in series with responses $h_1[n]$ and $h_2[n]$ without altering the overall input-output relationship (see figure 4-6).

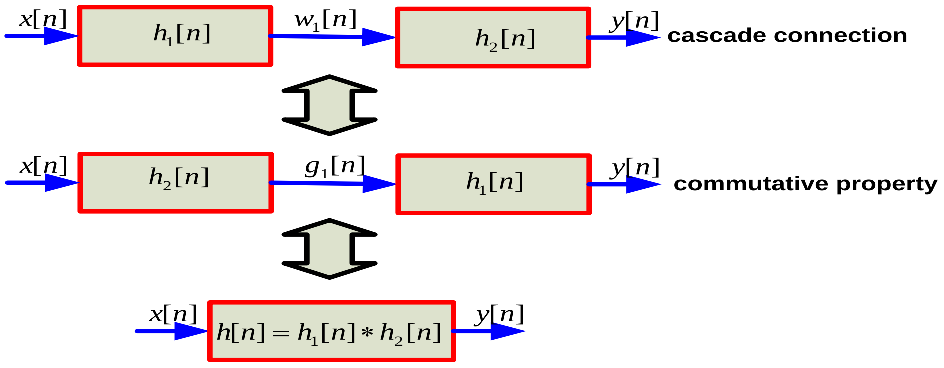


Figure 4-6: Interpretation of the associative and commutative property of convolution

3. Distributive over signal addition

$$x[n] * h_1[n] + x[n] * h_2[n] = x[n] * \left(h_1[n] + h_2[n]\right)$$

This equation can be interpreted as the following:

For two LTI systems with impulse responses $h_1[n]$ and $h_2[n]$ excited by the same input signal $x[n]$, the sum of two responses is identical to the response of an overall system with impulse response

$$h[n] = h_1[n] + h_2[n]$$

The distributive property implies that parallel interconnection of two LTI system is an LTI system with impulse response as sum of two impulse responses (see figure 4-7).

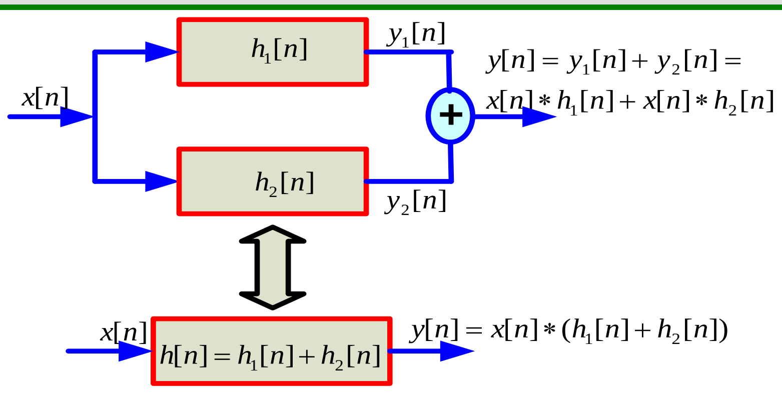


Figure 4-7: Interpretation of the distributive property of convolution

The generalization of distributive property to more than two systems in parallel follows directly by mathematical induction as the following

$$h[n] = \sum_{k=1}^{L} h_k[n]$$

4. **The identity sequence:** the unit sequence $\delta[n]$ is the identity element for convolution, that is

$$x[n] * \delta[n] = x[n]$$

5. **Delay operation:** if we shift $\delta[n]$ by k, the convolution signal is also shifted by k, that is

$$x[n] * \delta[n-k] = x[n-k]$$

6. **Multiplication by a constant**

$$\left(ax[n]\right) * h[n] = a\left(x[n] * h[n]\right)$$